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Integral Representation of the generalized K-Function

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ABSTRACT

This paper deals with the integral representation and fractional calculus of the generalized K-function. Several special cases have also been discussed and the generalized M-series which is introduced by Sharma and Jain.

Keywords: Integral, K-Function, Arbitrary Orders, Power Series.

1. Introduction

Fractional calculus deals with derivatives and integrals of arbitrary orders. During the last three decades fractional calculus has been applied to almost every field of mathematics like special functions, science, engineering and technology. The Mittag-Leffler function has gained importance and popularity during the last one decade due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. This function is introduced and studied by the Swedish mathematician Gosta Mittag-Leffler [11, 12] in terms of the power series.

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+1)},\ (\alpha>0) \qquad (1.1)$$

In 1905, a generalization of this series in the following form

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+\beta)} \qquad ,(\alpha,\beta>0) \qquad (1.2)$$

2. The integral representation of the generalized K-function:

In this section, we derive formulae based on integral representations of the K-function [9]. The results presented in the form of the theorems are given below:

Theorem 1. If $\alpha,\beta,\gamma,\delta\in C,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma),\beta>\alpha>0$ then

$$ {}_{p}K_{q}^{\alpha,\beta;\gamma,\delta}(z) = k\, z^{\alpha-\beta} \int_0^\infty e^{\left(-\frac{t^k}{z^k}\right)} t^{\beta-\alpha-1} \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{t^n}{(\delta)_n \Gamma(\alpha n+\beta) \Gamma\left(\frac{\beta-\alpha+n}{k}\right)} dt $$

(1.3)

Where ${}_{p}K_{q}^{\alpha,\beta;\gamma,\delta}(z)$ is the generalized K-function.

Proof: On taking the term

$$ \int_0^\infty e^{\left(-\frac{t^k}{z^k}\right)} t^{\beta-\alpha-1} \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{t^n}{(\delta)_n \Gamma(\alpha n+\beta) \Gamma\left(\frac{\beta-\alpha+n}{k}\right)} dt $$

(1.4)

Now interchanging the order of summation and integration which is permissible under the stated conditions and putting $\frac{t^k}{z^k} = u$, we get

$$ = \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^{\beta-\alpha+n}}{(\delta)_n \Gamma(\alpha n+\beta) \Gamma\left(\frac{\beta-\alpha+n}{k}\right)} \frac{1}{k} \int_0^\infty e^{-u} u^{\frac{\beta-\alpha+n}{k}-1} dt $$

(1.5)

On applying $ {}_{c}D_{x}^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t) dt $ (1.6)

It reduces to

$$ = \frac{z^{\beta-\alpha+n}}{k} {}_{p}K_{q}^{\alpha,\beta;\gamma,\delta}(z) $$

(1.7)

Therefore

$$ {}_{p}K_{q}^{\alpha,\beta;\gamma,\delta}(z) = k\, z^{\alpha-\beta} \int_0^\infty e^{\left(-\frac{t^k}{z^k}\right)} t^{\beta-\alpha-1} \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{t^n}{(\delta)_n \Gamma(\alpha n+\beta) \Gamma\left(\frac{\beta-\alpha+n}{k}\right)} dt $$

(1.8)

This completes the proof.

Theorem 2 If $\alpha, \beta, \gamma, \delta \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma), \beta > \alpha > 0$ then

$$ {}_pK_q^{\alpha,\beta;\gamma,\delta}(z) = \frac{1}{\alpha\Gamma(\beta-\alpha)} \int_0^1 (1-t^{\frac{1}{\alpha}})^{\beta-\alpha-1} \, {}_pK_q^{\alpha,\alpha;\gamma,\delta}(tz)dt $$

(1.9)

Where ${}_pK_q^{\alpha,\beta;\gamma,\delta}(z)$ is the generalized K-function given by (1.6).

Proof: Consider

$$ \int_0^1 (1-t^{\frac{1}{\alpha}})^{\beta-\alpha-1} \, {}_pK_q^{\alpha,\alpha,\gamma,\delta}(tz)dt \, . $$

(1.10)

On interchanging the order of summation and integration which is permissible under the stated conditions, using (1.6) and substituting $t^{\frac{1}{\alpha}} = u$, we obtain

$$ = \alpha \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{(\delta)_n \Gamma(\alpha n + \beta)} \int_0^1 u^{\alpha n+\alpha-1}(1-u)^{\beta-\alpha-1} du $$

(2.1)

On evaluating the inner integral with the help of Beta function, we arrive at

$$ = \alpha\Gamma(\beta-\alpha) \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{(\delta)_n \Gamma(\alpha n + \beta)} $$

(2.2)

Again using (1.6), it reduces to

$$ = \alpha\Gamma(\beta-\alpha) \, {}_pK_q^{\alpha,\beta;\gamma,\delta}(z) \cdot $$

(2.3)

Therefore

$$ {}_pK_q^{\alpha,\beta;\gamma,\delta}(z) = \frac{1}{\alpha\Gamma(\beta-\alpha)} \int_0^1 (1-t^{\frac{1}{\alpha}})^{\beta-\alpha-1} \, {}_pK_q^{\alpha,\alpha;\gamma,\delta}(tz)dt $$

(2.4)

This proves theorem

Corollary 1. If $\alpha, \beta \in C, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \beta > \alpha > 0$ then

$$ {}_{p}\overset{\alpha,\beta}{M}_{q}(z) = k\, z^{\alpha-\beta} \int_0^{\infty} e^{\left(-\frac{t^k}{z^k}\right)} t^{\beta-\alpha-1} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{t^n}{\Gamma(\alpha n+\beta)\Gamma\left(\frac{\beta-\alpha+n}{k}\right)} dt $$

(2.5)

Where ${}_{p}\overset{\alpha,\beta}{M}_{q}(z)$ is the generalized M-series.

Corollary 2.5 If $\alpha, \beta \in C, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \beta > \alpha > 0$ then

$$ {}_{p}\overset{\alpha,\beta}{M}_{q}(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\, {}_{p}\overset{\alpha,\beta-\alpha}{M}_{q}[z(1-t)^{\alpha}]dt $$

(2.6)

Where ${}_{p}\overset{\alpha,\beta}{M}_{q}(z)$ is the generalized M-series.

Solutions of the equations and problems considered were obtained in terms of generalized Mittag-Leffler functions.

Special Cases:

If we set $\delta = \gamma = 1$ in theorems (2.2) and (2.3), leads to the integral representation of generalized M-series introduced by Sharma and Jain.

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